Exercise 1

Find the series solution for the following homogeneous second order ODEs:

$$u'' + xu' + u = 0$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \to \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \to \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Now we substitute these series into the ODE.

$$u'' + xu' + u = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Since we want to combine the series, we want the first series to start from $n = 0$ just like the other two. We can start it at $n = 0$ as long as we replace n with $n + 2$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

The point of doing this is so that we have x^n in every series. Now we can combine the series.

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} x^n + n a_n x^n + a_n x^n] = 0$$

Factor the left side.

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+1) a_n] x^n = 0$$

Thus,

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$
$$(n+2)a_{n+2} + a_n = 0$$
$$a_{n+2} = -\frac{1}{n+2}a_n.$$

Now that we know the recurrence relation, we can determine the coefficients.

$$\begin{aligned}
n = 0: &\qquad a_2 = -\frac{a_0}{2} \\
n = 1: &\qquad a_3 = -\frac{a_1}{3} \\
n = 2: &\qquad a_4 = -\frac{a_2}{4} = \frac{a_0}{8} \\
n = 3: &\qquad a_5 = -\frac{a_3}{5} = \frac{a_1}{15} \\
n = 4: &\qquad a_6 = -\frac{a_4}{6} = -\frac{a_0}{48} \\
n = 5: &\qquad a_7 = -\frac{a_5}{7} = -\frac{a_1}{105} \\
\vdots &\qquad \vdots
\end{aligned}$$

Therefore,

$$u(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots\right) + a_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \cdots\right),$$

where a_0 and a_1 are arbitrary constants.